Compressive Sensing via Sparse Sensing

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Thanks: Sundeep Chepuri, Mario Coutiño, Guillermo Jimenez

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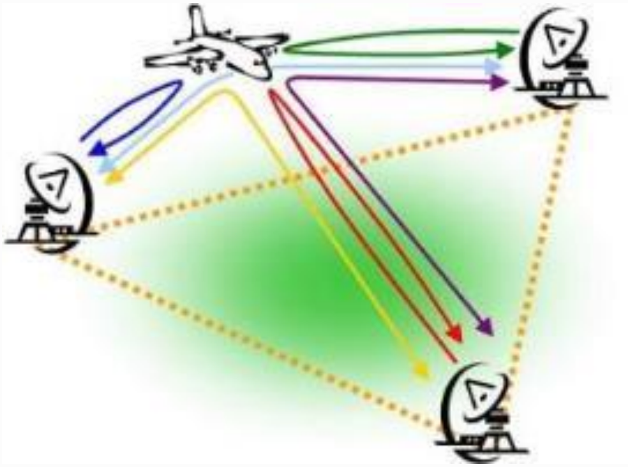
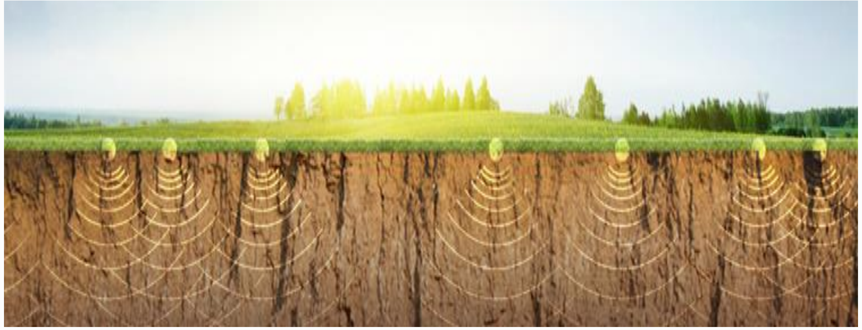
Motivation

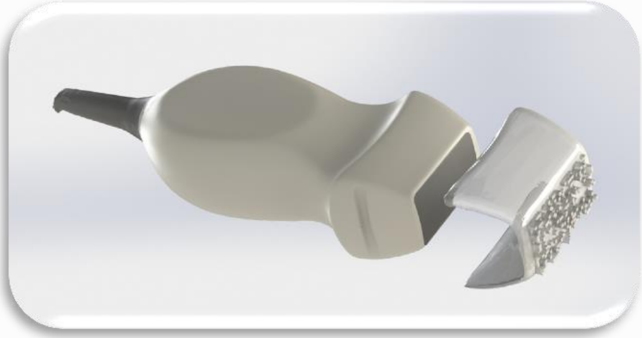


Distributed radar

Radio astronomy



Microseismic event detection



Ultrasound imaging

Problem statement

Design of structured (sparse) space-time samplers

or

sparse sensing design

The term “**sparse sensing = sampling**” has been used earlier:

- Sampling sparse signals *[Vetterli et al.-2008]*
- Covariance reconstruction and array processing *[Vaidyanathan et al.-2011]*

- T. Blu, P.L. Dragotti, M.Vetterli, P. Marziliano, and L. Coulot. “Sparse sampling of signal innovations,” *IEEE Signal Processing Magazine*, vol. 25, no. 2, pp. 31-40, Mar. 2008.
- P.P. Vaidyanathan and P. Pal. "Sparse sensing with co-prime samplers and arrays." *IEEE Transactions on Signal Processing,* vol. 59, no. 2, pp. 573-586, Feb. 2011.

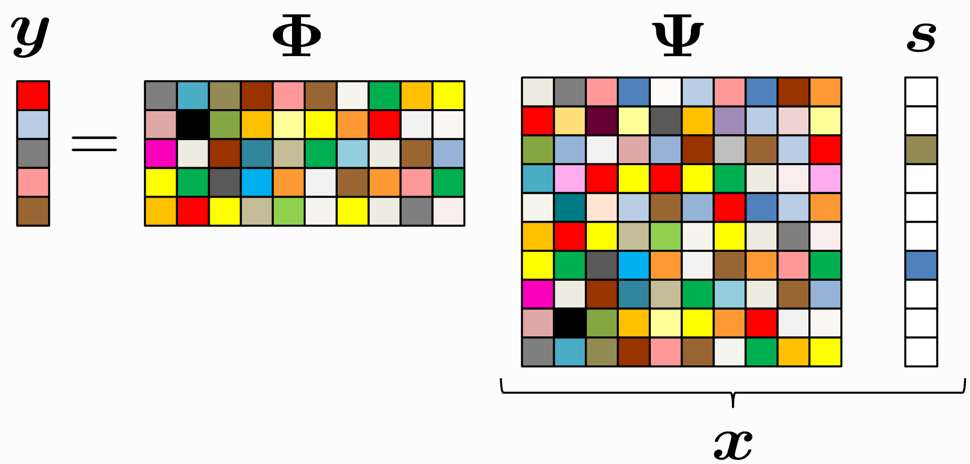
Why sparse sensing?

- **Economical** constraints (hardware cost)
- Limited **physical space**
- Limited data **storage space**
- Reduce **communications bandwidth**
- Reduce **processing overhead**

Compressive sensing

- **State-of-the-art tool** for sensing cost reduction

[Donoho 2006], [Candès 2006]



- Random linear projections of Nyquist rate samples
- Sparse signal reconstruction

$$\min_{\boldsymbol{s}} \|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{\Psi}\boldsymbol{s}\|_2^2 \quad \text{s.to} \quad \|\boldsymbol{s}\|_0 \leq \epsilon$$

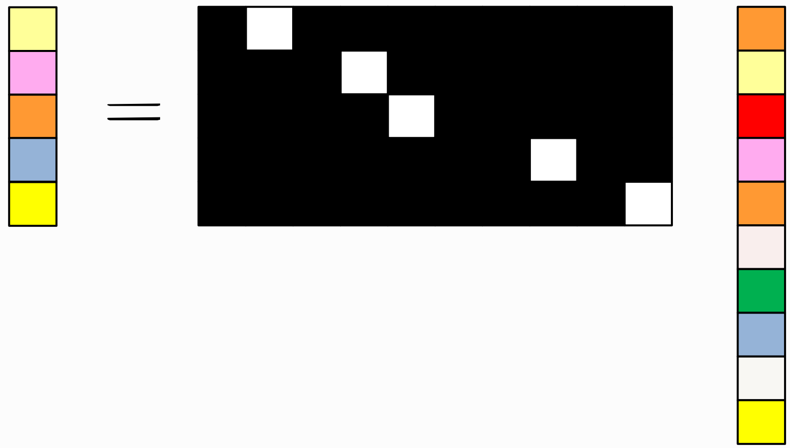
- D. L. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory*, vol. 52, no. 4, pp. 1289–1306, Apr. 2006.
- E. J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. Inf. Theory*, vol. 52, no. 2, pp. 489–509, feb. 2006.

Sparse sensing vs. compressed sensing

	Compressed sensing	Sparse sensing
Sparse signal	**needed**	**not needed**
Samplers	**random**	**deterministic, sparse**
Compression	**robust**	**practical, controllable**
Signal processing task	**sparse signal reconstruction**	**any statistical inference**

Discrete sparse sensing

$$\boldsymbol{y} \qquad \boldsymbol{\Phi}(\boldsymbol{w}) \in \{0,1\}^{K\times M} \quad \boldsymbol{x}$$



- Candidate measurement set (samples, sensors, etc.):

$$\boldsymbol{x} = [x_1, x_2, \ldots, x_M]^T$$

- Sparse sensing vector (Boolean):

$$\boldsymbol{w} = [w_1, w_2, \ldots, w_M]^T \in \{0,1\}^M$$

Design problem

Select the “best” subset of sensors out of the candidate sensors that guarantee a certain desired inference performance.

Formulation 1

$$\begin{aligned} \arg\min_{\boldsymbol{w}} \quad & \|\boldsymbol{w}\|_0 \\ \text{s.to} \quad & f(\boldsymbol{w}) \leq \lambda \\ & \boldsymbol{w} \in \{0,1\}^M \end{aligned}$$

Formulation 2

$$\begin{aligned} \arg\min_{\boldsymbol{w}} \quad & f(\boldsymbol{w}) \\ \text{s.to} \quad & \|\boldsymbol{w}\|_0 = K \\ & \boldsymbol{w} \in \{0,1\}^M \end{aligned}$$

$f(\boldsymbol{w})$ inference performance metric

λ prescribed accuracy

K sample size

Nonconvex Boolean problem

Solutions to the combinatorial problem

Exact solutions:

$M = 30, 2^M = 10^9$

- Exhaustive search over
  - 2^M possible candidates for *formulation 1*
  - $\binom{M}{K}$ possible candidates for *formulation 2*

- Branch-and-bound methods

  [Lawler-Wood-1966], [Nguyen-Miller-1992]

  - long runtimes even for a modest sized problem

- E. L. Lawler and D. E. Wood, "Branch-and-bound methods: A survey," *Oper. Res.*, vol. 14, pp. 699–719, 1966.
- N. Nguyen and A. Miller, "A review of some exchange algorithms for constructing discrete D-optimal designs," *Comput. Statist. Data Anal.*, vol. 14, pp. 489–498, 1992

Solutions to the combinatorial problem

Suboptimal solutions:

- **Convex** optimization (polynomial time)

  [Joshi-Boyd-2009], [Chepuri-Leus-2015]

  - convex relaxation for $\|\boldsymbol{w}\|_0, \ \{0,1\}, f(\boldsymbol{w})$
  - thresholding, randomization to get back a Boolean solution
  - Semidefinite program (SDP) typically

- S. Joshi and S. Boyd, “Sensor selection via convex optimization,” *IEEE Trans. Signal Process.*, vol. 57, no. 2, pp. 451–462, Feb. 2009
- S.P. Chepuri and G. Leus. “Sparsity-Promoting Sensor Selection for Non-linear Measurement Models,” *IEEE Trans. on Signal Processing*, vol. 63, no. 3, pp. 684-698, Feb. 2015.

Solutions to the combinatorial problem

Suboptimal solutions:

- **Submodular** optimization (linear search)

  [Krause-Singh-Guestrin-2008], [Ranieri-Chebira-Vetteri-2014]

  - greedy search
  - solution is near optimal

- A. Krause, A. Singh, and C. Guestrin, “Near-optimal sensor placements in Gaussian processes: Theory, efficient algorithms and empirical studies,” *J. Machine Learn. Res.*, vol. 9, pp. 235–284, Feb. 2008.
- J. Ranieri, A. Chebira, and M. Vetterli, “Near-optimal sensor placement for linear inverse problems,” *IEEE Trans. Signal Process.*, vol. 62, no. 5, pp. 1135–1146, Mar. 2014

Convex optimization

Requires $f(\cdot)$ to be convex function

- Boolean constraint is relaxed to the box constraint $[0,1]^M$
- ℓ_0(-quasi) norm is relaxed to either ℓ_1-norm: $\sum_{m=1}^{M} w_m$ or a form that is iteratively convex *[Candés-Wakin-Boyd-2008]*

Formulation 1

$$\arg\min_{\boldsymbol{w}} \ \mathbf{1}^T \boldsymbol{w}$$

$$\text{s.to} \quad f(\boldsymbol{w}) \leq \lambda$$

$$\boldsymbol{w} \in [0,1]^M$$

Formulation 2

$$\arg\min_{\boldsymbol{w}} \ f(\boldsymbol{w})$$

$$\text{s.to} \quad \mathbf{1}^T \boldsymbol{w} = K$$

$$\boldsymbol{w} \in [0,1]^M$$

- E. Candés, M. Wakin, and S. Boyd, “Enhancing sparsity by reweighted minimization l1-minimization,” *J. Fourier Anal. Appl.*, vol. 14, pp. 877–905, 2008.

Submodular optimization

Requires $f(\cdot)$ to be submodular monotonically increasing function

- Define the sampling set:

$$\mathcal{X} := \mathcal{S} = \{m | w_m = 1, m = 1, 2, \ldots, M\}$$

or $$\mathcal{X} := \mathcal{M} \backslash \mathcal{S} = \{m | w_m = 0, m = 1, 2, \ldots, M\}$$

- *Set* function $f(\mathcal{X})$ is submodular, if $\forall \mathcal{X} \subseteq \mathcal{Y} \subset \mathcal{M},\ s \in \mathcal{M} \backslash \mathcal{Y}$

$$f(\mathcal{X} \cup \{s\}) - f(\mathcal{X}) \geq f(\mathcal{Y} \cup \{s\}) - f(\mathcal{Y})$$

- If $f(\mathcal{X})$ is monotonically increasing, i.e., $f(\mathcal{X} \cup \{s\}) \geq f(\mathcal{X})$

Submodular optimization

Algorithm 1 Greedy algorithm

1. **Require** $\mathcal{X} = \emptyset, L$.
2. **for** $k = 1$ to L
3. $\quad s^* = \arg\max_{s \notin \mathcal{X}} f(\mathcal{X} \cup \{s\})$
4. $\quad \mathcal{X} \leftarrow \mathcal{X} \cup \{s^*\}$
5. **end**
6. **Return** $\mathcal{X}$

Linear search

$$L = K \text{ or } L = M - K$$

Then, greedy algorithm is near-optimal

$$f(\mathcal{X}) \geq \underbrace{(1 - 1/e)}_{63\%} \max_{|\mathcal{Y}|=L} f(\mathcal{Y})$$

[Nemhauser-Wolsey-Fisher-1978]

- G. L. Nemhauser, L. A. Wolsey, and M. L. Fisher, "An analysis of approximations for maximizing submodular set functions— I," Mathematical Programming, vol. 14, no. 1, pp. 265–294, 1978.

Sparse sensing for estimation

- Unknown parameter vector $\boldsymbol{\theta} \in \mathbb{C}^N$ follows

$$y_m = w_m \overbrace{h_m(\boldsymbol{\theta}, n_m)}^{x_m \sim p_m(x;\boldsymbol{\theta})}, \; m = 1, 2, \ldots, M$$

- Linear observations with indep. additive Gaussian noise

[Joshi-Boyd-09]

- What about more general cases?
- Exact MSE is hard to optimize and depends on algorithm
- Use the **Cramér-Rao bound** as the performance metric

$$\mathbb{E}\{(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T\} \geq \boldsymbol{C} = \boldsymbol{F}^{-1}$$

Fisher Information matrix (FIM)

- S. Joshi and S. Boyd, "Sensor Selection via Convex Optimization," *IEEE Transactions on Signal Processing*, vol. 57, no. 2, pp. 451–462, Feb. 2009

Statistically independent observations

- Independent observations: FIM is additive

$$\boldsymbol{F}(\boldsymbol{w},\boldsymbol{\theta}) = \sum_{m=1}^{M} w_m \boldsymbol{F}_m(\boldsymbol{\theta})$$

- Dependent Gaussian observations: $\boldsymbol{x} \sim \mathcal{N}\left(\boldsymbol{h}(\boldsymbol{\theta}), \boldsymbol{\Sigma}\right)$

$$\boldsymbol{F}(\boldsymbol{w},\boldsymbol{\theta}) = \left[\boldsymbol{\Phi}(\boldsymbol{w})\boldsymbol{J}(\boldsymbol{\theta})\right]^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{w}) \left[\boldsymbol{\Phi}(\boldsymbol{w})\boldsymbol{J}(\boldsymbol{\theta})\right]$$

$$\boldsymbol{J}(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{h}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \qquad \boldsymbol{\Sigma}^{-1}(\boldsymbol{w}) = \Big(\boldsymbol{\Phi}(\boldsymbol{w})\boldsymbol{\Sigma}\boldsymbol{\Phi}^T(\boldsymbol{w})\Big)^{-1}$$

Use $\boldsymbol{\Sigma} = a\boldsymbol{I} + \boldsymbol{S}$ with $a > 0$ and $\boldsymbol{S} \succ \boldsymbol{0}$ to obtain proxy

$$\begin{bmatrix} \boldsymbol{S}^{-1} + a^{-1}\mathrm{diag}(\boldsymbol{w}) & \boldsymbol{S}^{-1}\boldsymbol{J}(\boldsymbol{\theta}) \\ \boldsymbol{J}^T(\boldsymbol{\theta})\boldsymbol{S}^{-1} & \boldsymbol{J}^T(\boldsymbol{\theta})\boldsymbol{S}^{-1}\boldsymbol{J}(\boldsymbol{\theta}) \end{bmatrix}$$

This is again additive

$f(\boldsymbol{w})$ for estimation – scalar measures

- Prominent scalar measures
  - **E-optimality** measure (worst case error)

$$f(\boldsymbol{w}) := \lambda_{\mathsf{max}}\{\boldsymbol{F}^{-1}(\boldsymbol{w}, \boldsymbol{\theta})\} \longrightarrow \text{Convex (SDP)}$$

  - **A-optimality** measure (average error)

$$f(\boldsymbol{w}) := \mathsf{tr}\{\boldsymbol{F}^{-1}(\boldsymbol{w}, \boldsymbol{\theta})\} \longrightarrow \text{Convex (SDP)}$$

  - **D-optimality** measure (error volume)

$$f(\boldsymbol{w}) := \ln\det\{\boldsymbol{F}^{-1}(\boldsymbol{w}, \boldsymbol{\theta})\}$$

Convex

Submodular

  - Frame potential (approx. error) $\longrightarrow$ Submodular

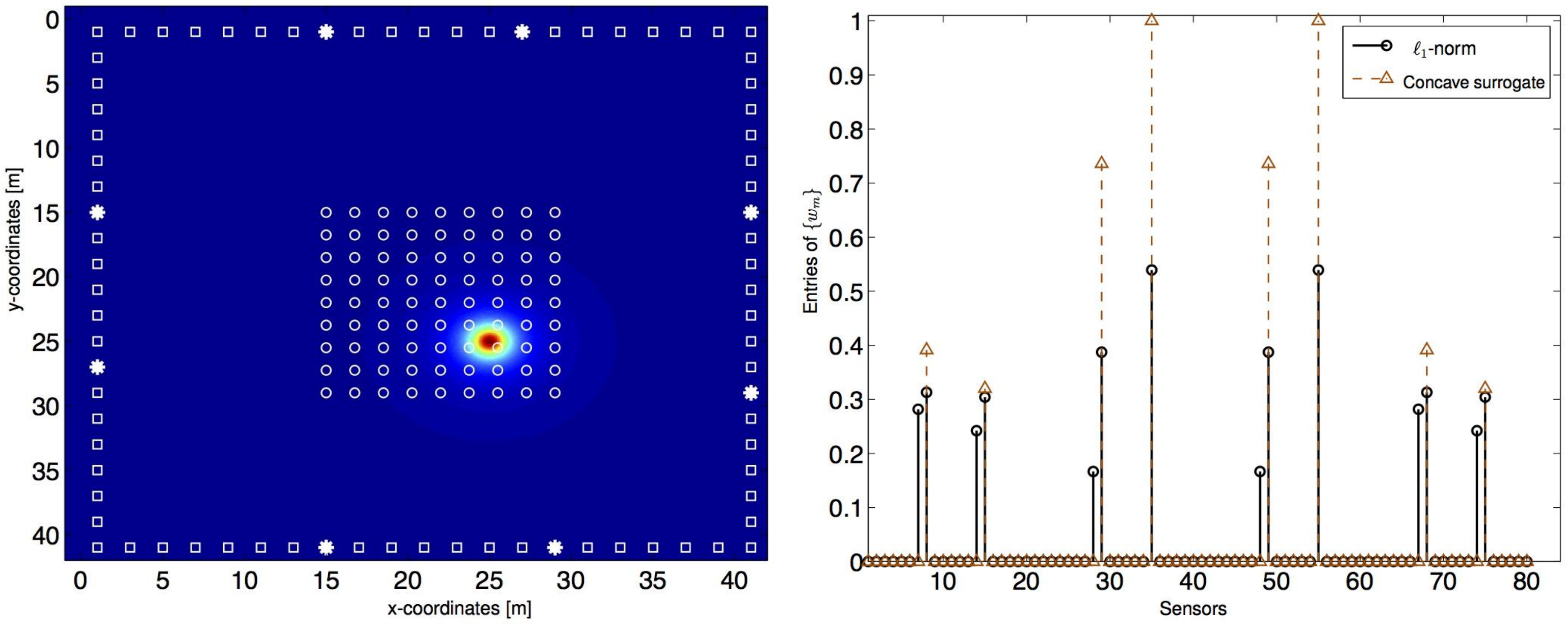
Sparse sensing for estimation

Setting	Convex	Submodular
Optimization criterion	Trace of inverse FIM = MSE Mimimum eigenvalue FIM	Logdet of FIM Frame potential
Independent Gaussian observations, linear	**SDP** using LMI	**Greedy** method
Independent observations, nonlinear	**SDP** - One LMI per possible solution - Single LMI for Bayesian cost	**Greedy** on Bayesian cost
Dependent Gaussian observations, linear	**SDP** using extended LMI	**Greedy** on extended Logdet of FIM
Dependent Gaussian observations, nonlinear	**SDP** - One ext. LMI per possible solution - Single ext. LMI for Bayesian cost	**Greedy** on Bayesian extended Logdet of FIM

- S.P. Chepuri and G. Leus, “Sparsity-Promoting Sensor Selection for Non-linear Measurement Models,” *IEEE Trans. on Signal Processing*, vol. 63, no. 3, pp. 684-698, Feb. 2015.
- S. Liu, S.P. Chepuri, M. Fardad, E. Masazade, G. Leus, and P.K. Varshney, “Sensor Selection for Estimation with Correlated Measurement Noise,” *IEEE Transactions on Signal Processing*, Mar. 2016.
- S. Rao, S.P. Chepuri, and G. Leus, “Greedy Sensor Selection for Non-Linear Models,” In *Proc. to the IEEE Workshop on Comp. Adv. in Multi-Sensor Adaptive Proc.* (CAMSAP 2015), Cancun, Mexico, December 2015.

Example: RSS target localization

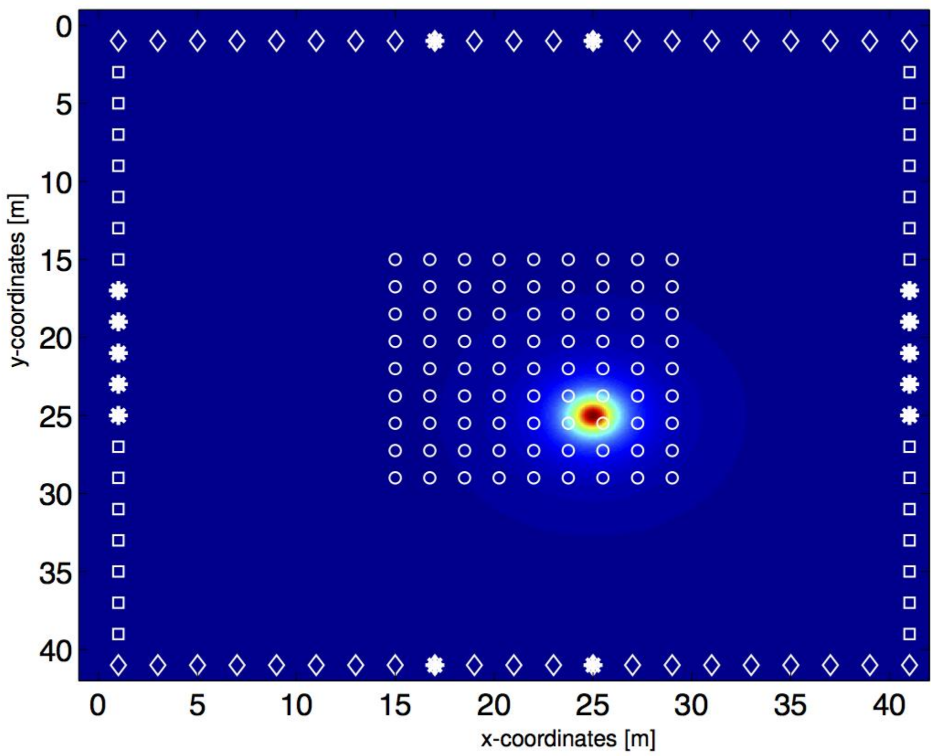
- Sensors along all edges are not correlated

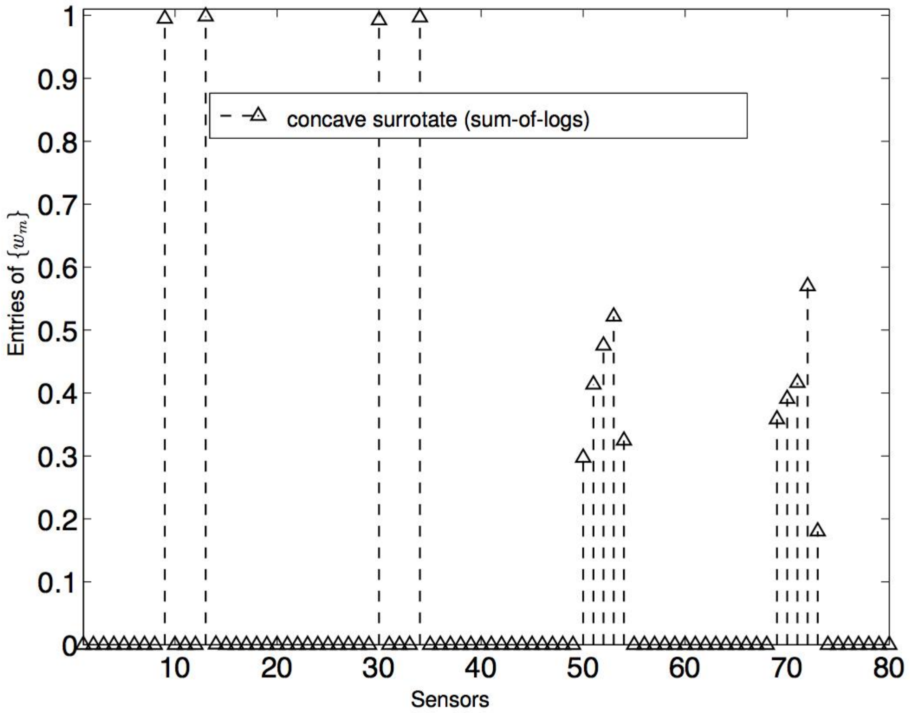


Out of 80 available access point locations, 8 access points are selected.

Example: RSS target localization

- Sensors along horizontal edges are equicorrelated (correlation coefficient 0.5)
- Sensors along vertical edges are not correlated





Out of 80 available uncorrelated and correlated access point locations, 14 access points are selected.

Sparse sensing for detection

- Observations follow (binary hypothesis testing)

$$\mathcal{H}_0 : \quad y_m = w_m x_m; x_m \sim p_m(x|\mathcal{H}_0),\ m = 1, 2, \ldots, M$$
$$\mathcal{H}_1 : \quad y_m = w_m x_m; x_m \sim p_m(x|\mathcal{H}_1),\ m = 1, 2, \ldots, M$$

- Independent Gaussian observations

[Cambanis-Masry-83], [Yu-Varshney-97], [Bajovic-Sinopoli-Xavier-11]

- What about more general cases?

- S. Cambanis and E. Masry, “Sampling designs for the detection of signals in noise,” *IEEE Trans. Inf. Theory*, vol. 29, no. 1, pp. 83–104, Jan. 1983.
- C.-T. Yu and P. K. Varshney, “Sampling design for Gaussian detection problems,” *IEEE Trans. Signal Process.*, vol. 45, no. 9, pp. 2328–2337, 1997.
- D. Bajovic, B. Sinopoli, and J. Xavier, “Sensor selection for event detection in wireless sensor networks,” *IEEE Trans. Signal Process.*, vol. 59, no. 10, pp. 4938–4953, Oct. 2011.

Sparse sensing for detection

Neyman Pearson setting

$$\arg \min_{\boldsymbol{w} \in \{0,1\}^M} \|\boldsymbol{w}\|_0$$
$$\text{s.to} \quad P_f(\boldsymbol{w}) \leq \alpha,$$
$$P_m(\boldsymbol{w}) \leq \beta$$

$$P_m = 1 - P(\widehat{\mathcal{H}} = \mathcal{H}_1 | \mathcal{H}_1)$$
$$P_f = P(\widehat{\mathcal{H}} = \mathcal{H}_1 | \mathcal{H}_0)$$

Bayesian setting

$$\arg \min_{\boldsymbol{w} \in \{0,1\}^M} \|\boldsymbol{w}\|_0$$
$$\text{s.to} \quad P_e(\boldsymbol{w}) \leq e$$

π_0, π_1 prior probabilities

$$P_e = \pi_0 P_f + \pi_1 P_m$$

- Exact error probabilities hard to optimize
- Seek weaker performance measures
  - Kullback-Leibler distance
  - J-divergence
  - Bhattacharyya distance

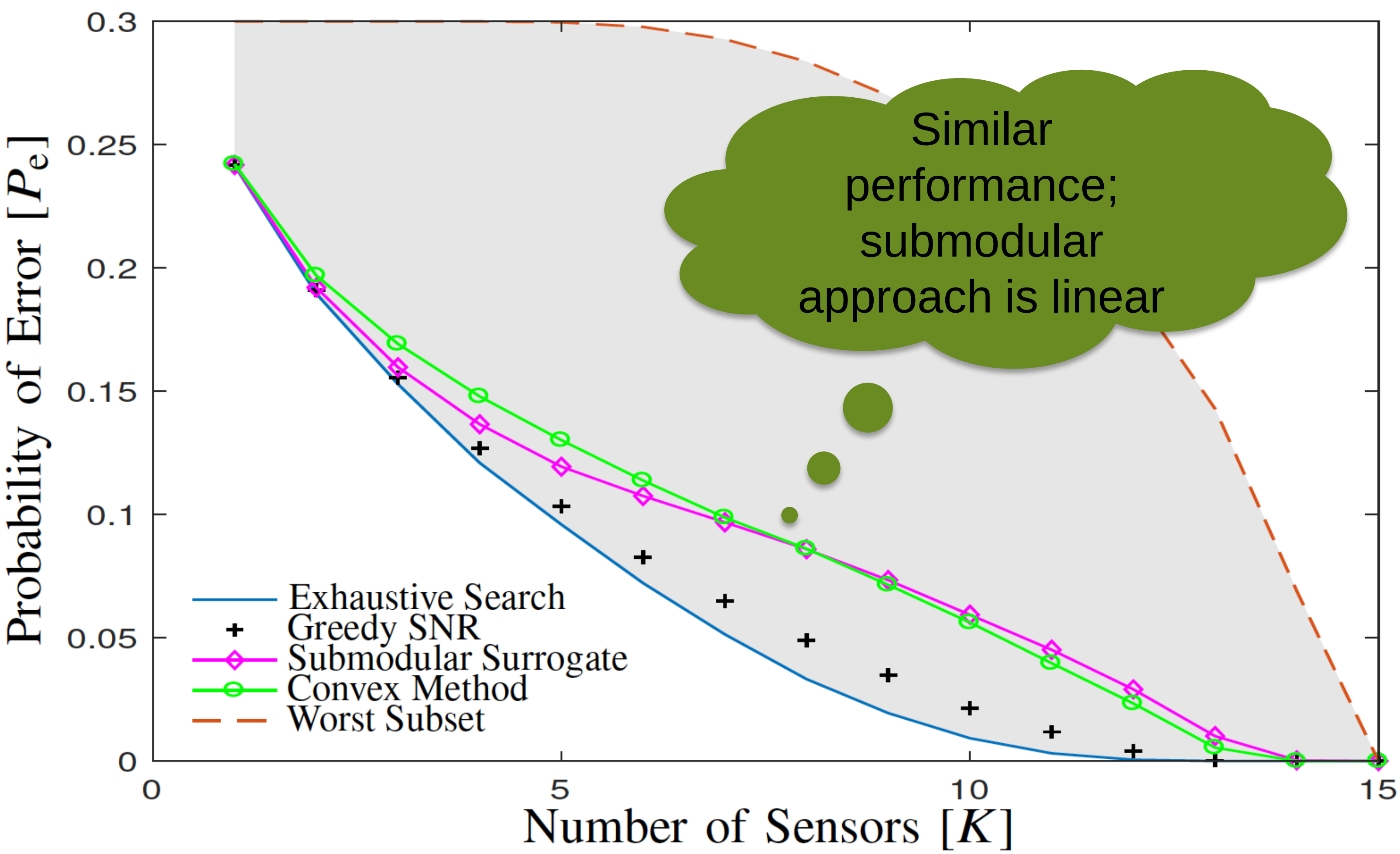
Sparse sensing for detection

Setting	Neyman-Pearson	Bayesian
Optimization criterion	Kullback-Leibler distance or J-divergence	Bhattacharyya distance or J-divergence
Independent observations	**Ordering** distances	**Ordering** distances
Dependent Gaussian observations (uncommon means)	**Convex:** SNR matrix **Greedy:** Logdet SNR matrix	**Convex:** SNR matrix **Greedy:** Logdet SNR matrix
Dependent Gaussian observations (uncommon variances)	**Sup-sub**: Kullback-Leibler dist. **Convex:** J-divergence	**Sup-sub**: Bhattacharyya dist. **Convex**: J-divergence

- M. Coutino, S.P. Chepuri, and G. Leus, "Near-Optimal Sparse Sensing for Gaussian Detection with Correlated Observations," *IEEE Transactions on Signal Processing*, vol 66, no. 15, pp. 4025-4039, Aug. 2018.
- S.P. Chepuri and G. Leus, "Sparse Sensing for Distributed Detection," *IEEE Trans. on Signal Processing*, vol. 16, no. 6, pp. 1446-1460, Mar. 2016.
- S.P. Chepuri and G. Leus, "Sparse Sensing for Distributed Gaussian Detection," In *Proc. of the International Conference on Acoustics, Speech, and Signal Processing* (ICASSP 2015), Brisbane, Australia, April 2015. (ICASSP best student paper award)

Example: Convex or submodular?

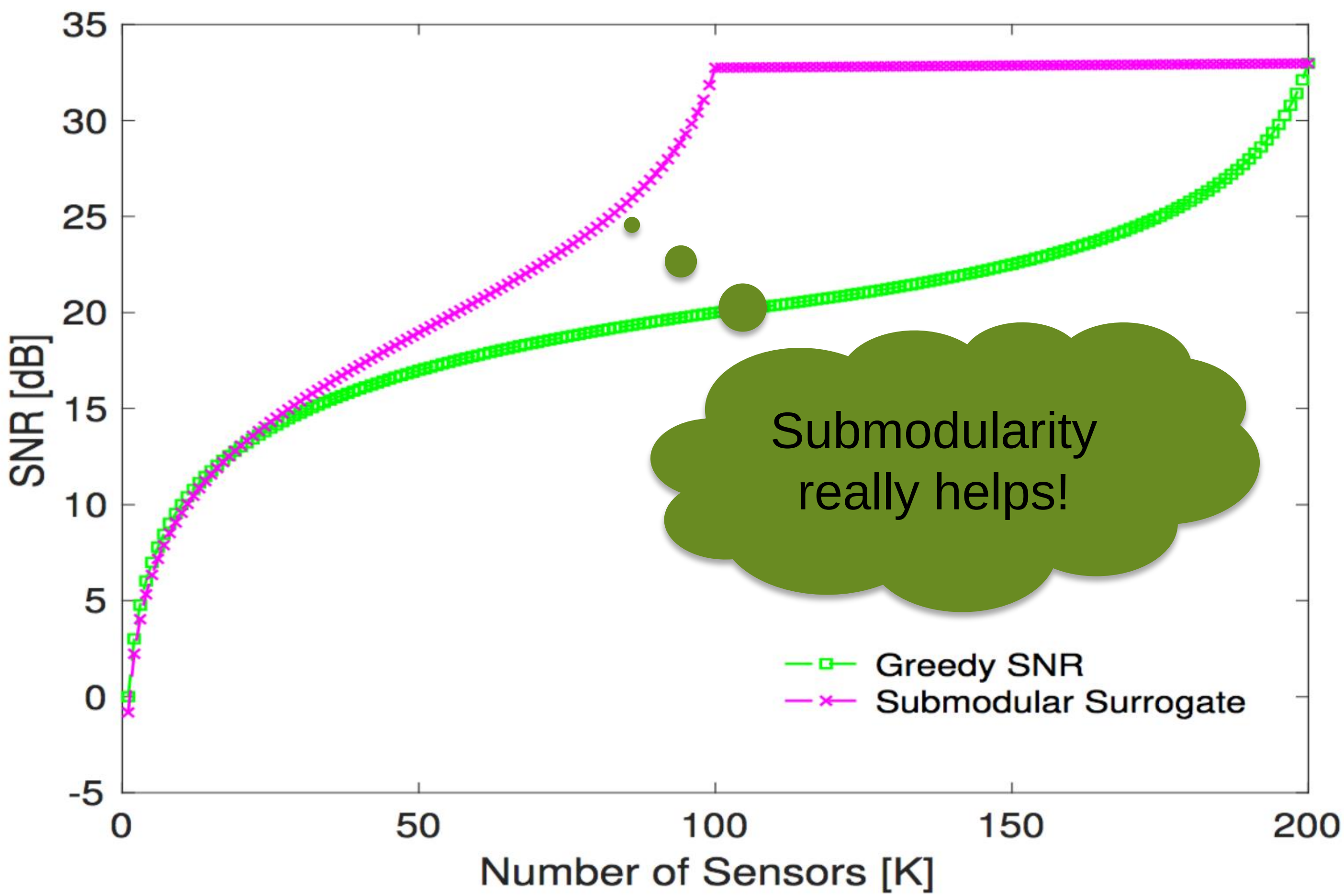
Random Toeplitz correlation matrices based on array processing



Example: Can “simple” greedy fail?

Subset of calibrated sensors

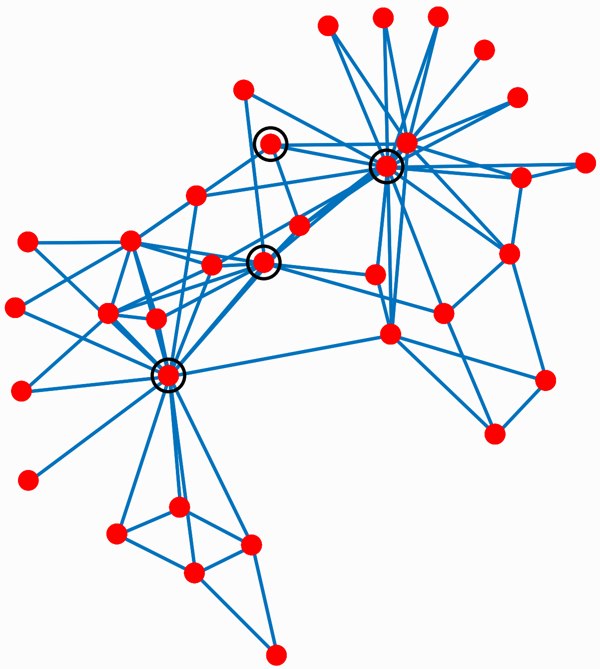
$$\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \boldsymbol{T} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \qquad \boldsymbol{T} = \mathrm{Toeplitz}([1, \rho^1, \rho^2, \ldots, \rho^{\mathrm{M}/2-1}])$$



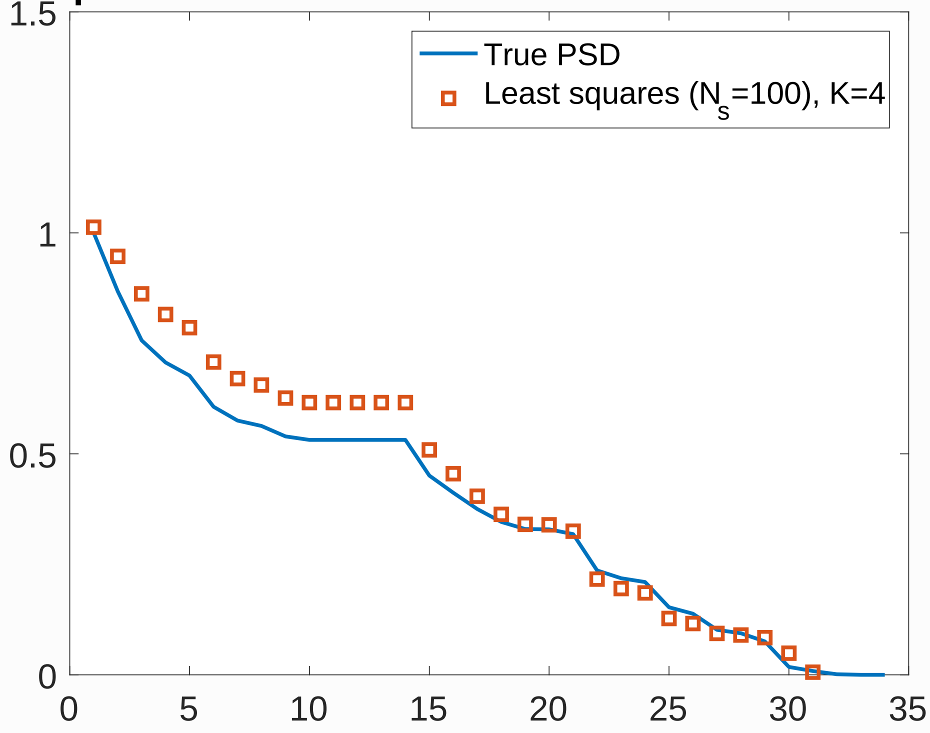
Recent developments

Subsampling signals on graphs

Sample 4 out of 34 nodes



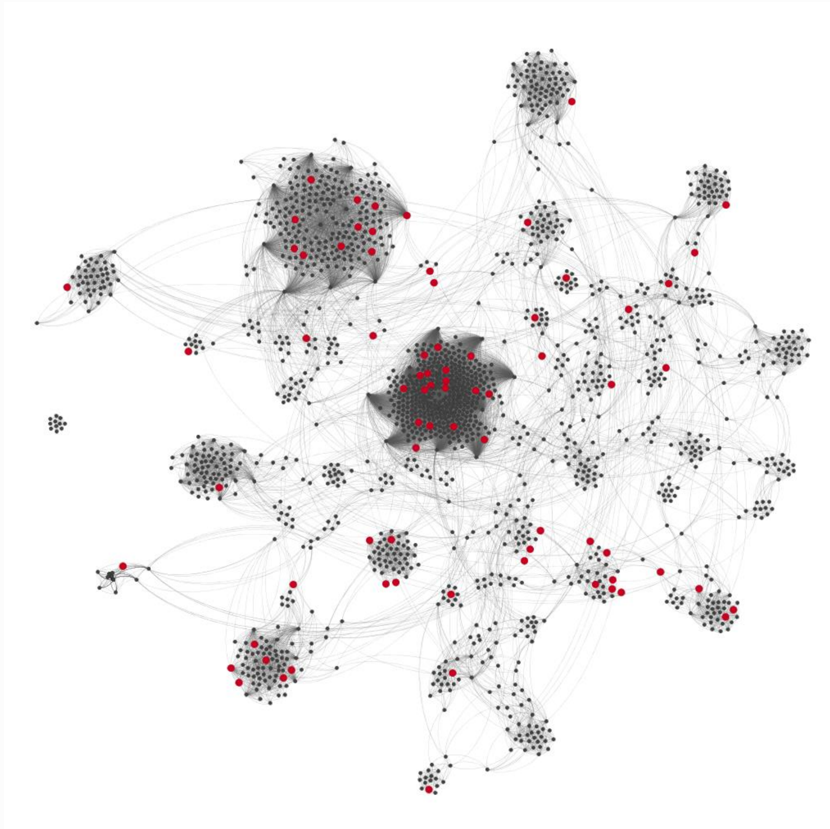
MA parametric PSD estimation from subset of nodes

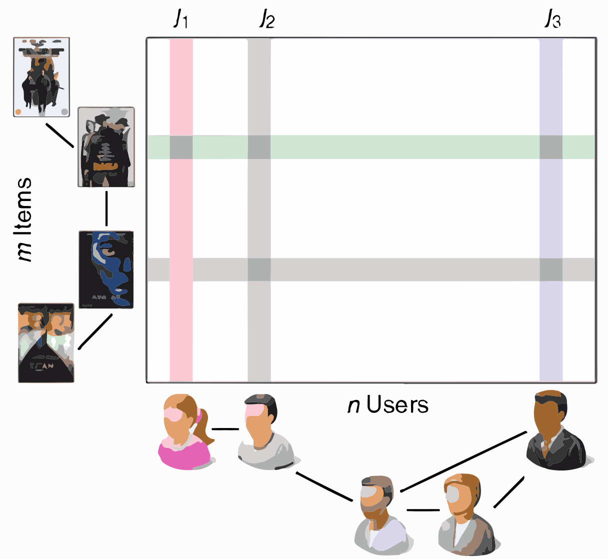


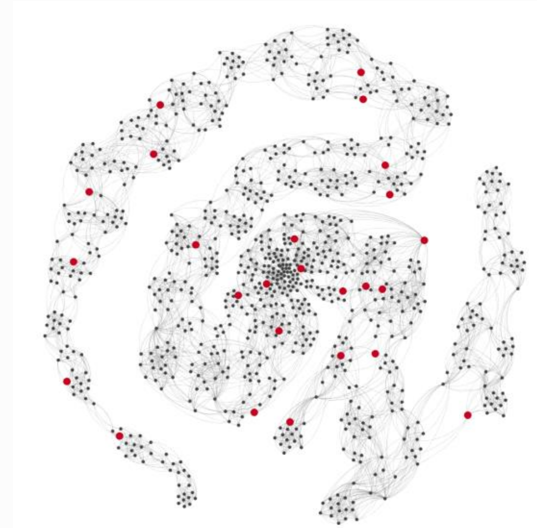
- S.P. Chepuri and G. Leus, "Graph Sampling for Covariance Estimation," *IEEE Jour. on Sel. Topics in Sig. Proc. and IEEE Trans. on Sig. and Info. Proc. over Networks, joint special issue on Graph Signal Processing*, vol. 3, no. 3, pp. 451-466, Sep. 2017.

Recent developments

Structured selection for tensors







Similar performance as state-of-the-art methods
but 1.875 measurements vs. 80.000 measurements

- G.Ortiz-Jimenez, M. Coutino, S.P. Chepuri, and G. Leus, “Sparse Sampling for Inverse Problems with Tensors,” *IEEE Transactions on Signal Processing*, submitted, Jun. 2018.

Conclusions

	Sparse sensing
Sparse signal	**not needed**
Samplers	**deterministic and sparse**
Compression	**practical, controllable**
Signal processing task	**any statistical inference**

- Design space-time sparse samplers

  Extend Nyquist-based classical sensing techniques

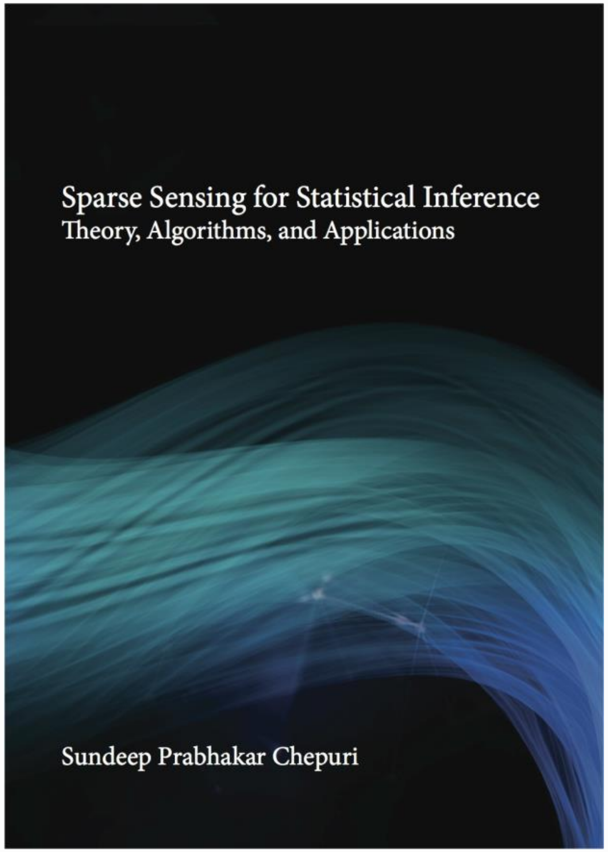
- Basic statistical inference problems

  Estimation, filtering, and detection

Reference material

PhD thesis

http://theses.eurasip.org/theses/648/sparse-sensing-for-statistical-inference-theory/



Monograph in Foundations and Trends in Signal Processing

Thank You!
Questions?