ON THE BIT DISTRIBUTION IN ASYMMETRIC MULTIPLE-DESCRIPTION CODING

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It has recently been shown that for asymmetric multiple-description lattice vector quantizers, given a fixed total bit budget, the optimal bit distribution among the descriptions is not unique but contains in fact a set of solutions, which all lead to minimal expected distortion. Contrary to ones intuition, this result indicates that distributing the bit budget such that most bits are transmitted on the bad channels yields equivalent performance as if most bits are transmitted on the good channels. In the present work we study some peculiar consequences of this result. Specifically, we consider minimization of an economical cost in addition to minimizing the expected distortion subject to a maximum bit budget. We also consider the related problem of minimizing the expected distortion subject to a maximum economical cost.

INTRODUCTION

Multiple-description coding (MDC) provides robustness against packet losses in packet switched networks like the Internet by distributing information about a source across several redundant descriptions. At the receiving side the source is reconstructed with a quality that depends upon the subset of received descriptions. A typical two-channel scheme is depicted in Fig. 1 where a total bit budget is split between the two descriptions, and the distortion observed at the receiver depends on which descriptions arrive. If both descriptions are received, the distortion, is lower than if only a single description is received.

The achievable rate-distortion region for the two-channel problem with respect to the Gaussian source and mean square error fidelity criterion has been known for more than two decades [1, 2]. It was, however, not until recently that schemes were introduced [3–17], that in the limit of infinite-dimensional source vectors approach the achievable region presented in [1, 2].

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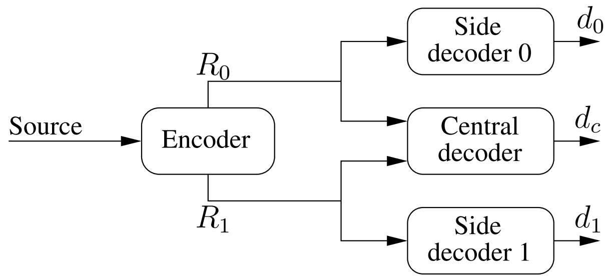

Figure 1: The traditional two-channel MDC scheme.

The schemes presented in [3–12] all exploit the idea of having only one central quantizer followed by an index-assignment algorithm that maps each central quantizer reconstruction point to reconstruction points in the side quantizers, an idea first presented by Vaishampayan [18]. The schemes presented in [13–17] are based on erasure codes and therefore fundamentally different from the quantizer based approaches.

Many existing schemes [3, 4, 7, 8, 10, 13–17] consider the symmetric case where the entropies of the side descriptions are equal and some [5, 6, 9, 11, 12] consider the asymmetric case where the entropies of the side descriptions are allowed to be unequal. Asymmetric schemes offer additional flexibility over symmetric schemes, since the bit distribution between descriptions is also a design parameter. In [5, 6, 9] optimal asymmetric two-channel multiple-description quantizers are derived subject to entropy constraints on the individual side entropies. However, since these schemes are subject to individual side entropy constraints and not subject to a single constraint on the sum of the side entropies, the problem of how to distribute a total bit budget among the two descriptions is not addressed. In the two-channel asymmetric scheme presented in [11] optimal multiple-description lattice vector quantizers (MD-LVQs) are designed subject to individual entropy constraints on side entropies and/or subject to a single entropy constraint on the sum of the side entropies. The extension of [11] to more than two channels is given in [12]. Both [11, 12] derive closed-form expressions for the optimal MD-LVQs which minimize the expected distortion based on the packet-loss probabilities of the channels. One of the key observations in [11, 12] is that for a given bit budget, the optimal bit distribution among the descriptions is not unique but contains in fact a set of solutions, which all lead to minimal expected distortion. Contrary to ones intuition, this result indicates that distributing the bit budget such that most bits are transmitted on the bad channels yields equivalent performance as if most bits are transmitted on the good channels! In the present work we study some peculiar consequences of this result. Specifically, we consider minimization of an economical cost in addition to minimizing the expected distortion subject to a maximum bit budget.

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PRELIMINARIES

In this section we introduce the notation that will be used in the paper and briefly describe the n-channel asymmetric MD-LVQ presented in [12].

Let $X \in \mathbb{R}^L$ be an arbitrary i.i.d. source and let $\Lambda \subset \mathbb{R}^L$ be a real lattice with Voronoi regions $V(\lambda), \lambda \in \Lambda$, given by

$$V(\lambda) \triangleq \{x \in \mathbb{R}^L : \|x-\lambda\|^2 \leq \|x-\lambda'\|^2, \forall \lambda' \in \Lambda\}, \tag{1}$$

where x is a realization of X and we define $\|x\|^2 \triangleq \frac{1}{L}x^T x$.

In [12] the central quantizer is a lattice $\Lambda_c \subset \mathbb{R}^L$ with Voronoi regions of volume ν. When a lattice Λ_c is used as a quantizer, a source x is mapped to the nearest lattice point or codeword λ_c in Λ_c. The side quantizers are sublattices $\Lambda_i \subset \Lambda_c$ of index $N_i, i = 0, \dots, K-1$, where $K > 0$ denotes the number of descriptions. The trivial case $K = 1$ leads to a single-description system, consisting only of the central quantizer.

A source X is quantized to a codeword λ_c by the central quantizer Λ_c. Hereafter follows an index assignment (mapping), which uniquely maps λ_c to a codeword λ_i in each of the side quantizers Λ_i. The codeword λ_i from the ith side quantizer is transmitted[1] on the ith channel. At the receiving side the source X is estimated based on the received codewords. If no descriptions are received X is estimated by its statistical mean, hence the distortion (per dimension) is equal to $E[\|X\|^2]$. The index assignment map is designed such that it is an invertible map in case all K descriptions are available, hence it is possible to obtain λ_c from the K-tuple $(\lambda_0, \dots, \lambda_{K-1})$.

Using standard high-resolution assumptions for lattice vector quantizers [19], the expected central distortion can be expressed as

$$d_c \approx G(\Lambda_c)\nu^{2/L}, \tag{2}$$

where $G(\Lambda_c)$ is the dimensionless normalized second moment of inertia [20] of the central quantizer. The minimum entropy R_c needed to achieve the central distortion d_c is given by [19]

$$R_c \approx h(X) - \frac{1}{L}\log_2(\nu), \tag{3}$$

where $h(X)$ is the component-wise differential entropy of the source. The side entropies are given by [12]

$$R_i \approx h(X) - \frac{1}{L}\log_2(N_i\nu). \tag{4}$$

[1] Quantization is followed by entropy coding and it is of course the index obtained from the entropy coder which is transmitted and not the codeword λ_i.

There are in general several ways of receiving κ out of K descriptions. Let $\mathcal{L}$ denote an index set consisting of all possible κ combinations out of $\{0,\dots,K-1\}$ so that $|\mathcal{L}| = \binom{K}{\kappa}$. We denote an element of $\mathcal{L}$ by $l = \{l_0,\dots,l_{\kappa-1}\}$. The complement l^c of l denotes the $K-\kappa$ indices not in l, i.e. $l^c = \{0,\dots,K-1\}\backslash l$. We will use the notation $\mathcal{L}_i$ to indicate the set of all $l\in\mathcal{L}$ that contains the index i, i.e., $\mathcal{L}_i = \{l : l\in\mathcal{L} \text{ and } i\in l\}$ and similar $\mathcal{L}_{i,j} = \{l : l\in\mathcal{L} \text{ and } i,j\in l\}$. Furthermore, let p_i be the packet-loss probability for the ith description and consequently let $\mu_i = 1-p_i$ be the probability that the ith description is received. Finally, let $p(l) = \prod_{i\in l}\mu_i\prod_{j\in l^c}p_j$, $p(\mathcal{L}) = \sum_{l\in\mathcal{L}}p(l)$, $p(\mathcal{L}_i) = \sum_{l\in\mathcal{L}_i}p(l)$ and $p(\mathcal{L}_{i,j}) = \sum_{l\in\mathcal{L}_{i,j}}p(l)$. For example for $K=3$ and $\kappa=2$ we have $\mathcal{L} = \{\{0,1\},\{0,2\},\{1,2\}\}$ and hence $p(\mathcal{L}) = \mu_0\mu_1p_2 + \mu_0\mu_2p_1 + \mu_1\mu_2p_0$.

Let $\hat{p}(\mathcal{L}) = \sum_{\kappa=1}^{K}p(\mathcal{L})$ and $\hat{\beta} = \sum_{\kappa=1}^{K}\beta$ where β depends on K and κ and is given by

$$\beta = \frac{1}{\kappa^2}\sum_{i=0}^{K-2}\sum_{j=i+1}^{K-1}\left(\frac{p(\mathcal{L}_i)p(\mathcal{L}_j)}{p(\mathcal{L})} - p(\mathcal{L}_{i,j})\right).$$

The total expected distortion is then, under high-resolution assumptions, given by [12]

$$d_a \approx G(\Lambda_c)2^{2(h(X)-R_c)}\hat{p}(\mathcal{L}) + \psi^{2/L}\hat{\beta}G(S_L)2^{2(h(X)-R_c)}2^{\frac{2K}{K-1}\left(R_c-\frac{1}{K}R_T\right)} + E[\|X\|^2]\prod_{i=0}^{K-1}p_i, \tag{5}$$

where $G(S_L)$ is the dimensionless normalized second moment of an L-sphere and ψ depends only on the number of descriptions K and the vector dimension L.

The optimal values of ν is given by [12]

$$\nu = 2^{L(h(X)-\frac{1}{K}R_T)}\left(\psi^{2/L}\frac{1}{K-1}\frac{G(S_L)}{G(\Lambda_c)}\frac{\hat{\beta}}{\hat{p}(\mathcal{L})}\right)^{\frac{L(K-1)}{2K}}, \tag{6}$$

and by letting $R_i = a_iR_T$ it can be shown [12] that

$$N_i = 2^{\frac{L}{K}(1-a_i)R_T}\left(\psi^{-2/L}(K-1)\frac{G(\Lambda_c)}{G(S_L)}\frac{\hat{p}(\mathcal{L})}{\hat{\beta}}\right)^{\frac{L(K-1)}{2K}}.$$

It follows from (3) and (4) that $R_c \geq a_iR_T$ so that $a_i \leq R_c/R_T$. In addition, since the rates must be positive, we obtain the following inequality

$$0 < a_iR_T \leq R_c, \quad i = 0,\dots,K-1. \tag{7}$$

From (5) we see that the expected distortion depends only on R_T and not the specific R_i values. Hence, the individual side entropies $R_i = a_iR_T$ can be arbitrarily chosen as long as a_i satisfy (7) and $\sum_i a_i = 1$.

OPTIMAL BIT DISTRIBUTION

In this section we consider minimization of an economical cost while maintaining minimum expected distortion subject to a maximum bit budget and we also consider the related problem of minimizing the expected distortion subject to a maximum economical cost. Other related problems like the duals of the previous mentioned problems are also easily solved with the methods presented in this section.

1) Minimizing expected distortion and economical cost: Consider IP-telephony applications which with the recent spread of broadband networks are being used extensively throughout the world today. Let us assume, for ease of presentation[2], that a user has access to two different channels both based on the unreliable user datagram protocol [21]. Channel 0 is a non priority-based channel whereas channel 1 is a priority-based channel or they are both priority-based channels but of different priorities. The priority-based channel favors packets with higher priority and the packet-loss probability p_1 on channel 1 is therefore lower than that of channel 0, i.e. $p_1 < p_0$. Assume the Internet telephony service provider (ITSP) in question charges a fixed amount of say $\$1$ ($\$2$) per bit transmitted via channel 0 (channel 1) and moreover assume that the total bit budget is limited to e.g. 6 bits (set by either the user, the application or the ITSP). The user is economically minded and wants the best quality that can be achieved while paying the least amount of money. It is tempting to transmit all the bits through channel 1 since it offers better quality than channel 0. However, as shown in [11], it is usually beneficial to make use of both channels. The importance of exploiting two channels is illustrated in Table 1 for the example given above for a total bit budget of 6 bits and packet-loss probabilities $p_0 = 5\%$ and $p_1 = 2\%$. The last column of Table 1 describes the expected distortion occurring when quantizing a Gaussian source. For further details on the exact setup of this experiment we refer the reader to [11]. Notice the peculiarity that since the total bit budget is limited to 6 bits then even if the user is willing to pay more than $\$8$ the performance would be no better than what can be achieved when paying exactly $\$8$.

2) Minimizing expected distortion subject to maximum economical cost: Let us assume that we have access to K different channels with packet-loss probabilities $p_i, i = 0, \dots, K-1$. Furthermore, assume that there is an economical cost associated with the available bandwidth of the individual channels so that channel i costs C_i dollars per bit/sec. Hence, if one decides to transmit at R_i bit/sec. it costs R_iC_i dollars. For example let us assume that $K = 3$ and that $C_0 = 2, C_1 = 4$ and $C_2 = 6$.

[2] With the n-channel scheme presented in [12] it is straightforward to extend the example to the case of $K > 2$ channels.

Network	R_0	R_1	Price	Quality	Expected distortion
Single-channel	6	0	\$6	Poor	-12.98 dB
Single-channel	0	6	\$12	Good	-16.91 dB
Two-channel	2	4	\$10	Optimal	-22.20 dB
Two-channel	**4**	**2**	**\$8**	**Optimal**	**-22.20 dB**

Table 1: A total bit budget of 6 bits is spent in four different ways. The bottom row shows the cheapest way of buying the 6 bits and still achieve optimal performance.

Let the packet-loss probabilities be $p_0 = 0.2, p_1 = 0.1$ and $p_2 = 0.3$. In the situation where we limit our financial budget $C_T = \sum R_i C_i$ to e.g. $C_T = 50$, it is clear that by using only channel 0 we can transmit at a rate of $R_0 = 25$ bit/sec., with a loss rate of 20%, whereas if we concentrate on channel 1 we can only transmit at half that rate i.e. $R_1 = 12.5$ bit/sec. but the loss rate is also halved. It is intriguing to ask the question, "How do we obtain the best performance that money can buy?", or in other words, how should the C_T dollars be spent such that the expected distortion is minimized.

Since we allow C_T to be arbitrarily split among any K' out of the K channels a brute-force combinatorial approach is impossible[3], unless we limit the number of feasible solutions (i.e. the number of triplets (C_0, C_1, C_2)) to a finite number. However, by doing so it is possible that the optimal solution is excluded from the set of feasible solutions. Instead we give a very simple solution to this problem by using the fact that given a total target entropy R_T, the bits can be arbitrarily split among K' descriptions, without performance loss, as long as the inequality (7) is satisfied for all $i = 0, \ldots, K' - 1$ and that $\sum a_i = 1$. It is also true that, for a fixed set of channels, a greater R_T leads to a lower expected distortion. Therefore, maximizing R_T for some K' channels will in fact minimize the expected distortion for these K' channels. In order to maximize R_T we should spend as much money as possible on the cheapest channels without violating (7). This should be done for all the $2^K - 1$ non-trivial subsets of the K channels, after which the optimal solution is easily found from that subset which yields the minimum expected distortion.

We now continue the example given above. The cheapest channels, i.e. channel 0 and 1 should be given the greatest amount of bits possible, hence $R_0 = R_1 = R_c$. For a given R_T the optimal ν is given by (6) which when inserting in (3) leads to

[3] Define the set of feasible solutions as $\mathcal{C} = \{(C_0, \ldots, C_{K-1}) : \sum R_i C_i = C_T$ and $(C_T/C_i)/R_T = a_i\}$ where $C_i \in \mathbb{R}^+, R_T = \sum C_T/C_i$ and a_i satisfies (7) and notice that $|\mathcal{C}| = \infty$.

$$
\begin{aligned}
R_c &\approx h(X) - \frac{1}{L}\log_2(\nu) \\
&= \frac{1}{3}R_T - \frac{1}{3}\log_2\left(\frac{1}{2}\psi^{2/L}\frac{G(S_L)}{G(\Lambda_c)}\frac{\hat{\beta}}{\hat{p}(\mathcal{L})}\right).
\end{aligned} \tag{8}
$$

Assuming optimal scalar quantizers and a unit-variance Gaussian source, it can be shown that $G(S_L)/G(\Lambda_c) = 1, \psi = 1.1547, h(X) = 2.0471, \hat{\beta} = 0.0551$ and $\hat{p}(\mathcal{L}) = 0.9940$ so that $R_c = 1/3(R_0 + R_1 + R_2) + 1.5862$. Since $R_c = R_0 = R_1$ we have that $R_0 = 1/3R_T + 1.5862 \Rightarrow R_0 = R_2 + 1.5862$. It is also true that

$$
R_0 = (C_T - R_1C_1 - R_2C_2)/C_0, \tag{9}
$$

from which we find the optimal bit-rates to be $R_0 = R_1 = 6.5460$ bit/sec. and $R_2 = 1.7874$ bit/sec. This gives a total bit-rate of $R_T = 14.8793$ bit/sec. and the expected distortion when inserted in (5) is $d_a = -21.8801$ [dB]. At this point we still need to evaluate the six remaining subsets of the K channels, i.e. $\{\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\}\}$ in order to find the optimal solution. This is done in Table 2 from which we conclude that, for this particular example, it is in fact optimal to make use of all three channels[4]. Notice that the best performance is *not* obtained when the total transmission rate is maximized, i.e. $R_T = 25$ bit/dim. but is in fact obtained for $R_T = 14.8793$ bit/dim.

Channels:	$\{0\}$	$\{1\}$	$\{2\}$	$\{0,1\}$	$\{0,2\}$	$\{1,2\}$	$\{\mathbf{0},\mathbf{1},\mathbf{2}\}$
R_0 [bit/sec.]	25	0	0	9.72	7.53	0	**6.55**
R_1 [bit/sec.]	0	12.5	0	7.64	0	6.24	**6.55**
R_2 [bit/sec.]	0	0	8.33	0	5.82	4.18	**1.79**
R_T [bit/sec.]	25	12.5	8.33	17.36	13.35	10.41	**14.88**
Exp. dist. [dB]	-6.99	-10.00	-5.23	-16.98	-12.21	-15.06	$\mathbf{-21.88}$

Table 2: A total amount of $50 is spent in various ways (see text for details).

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